

A new high precision data assimilation method for the barotropic primitive equation

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To overcome a lot of shortcomings of gradient computations in the variational data assimilation by the adjoint method, such as low accuracy, difficult implementation and great complexity et al. a new data assimilation method based on dual-number theory is proposed. The new data assimilation method can avoid the complicated reverse integration of the tangent-linear/adjoint model. It only need a forward integration in dual-number space to obtain the cost function and the corresponding gradient vector. The numerical experiment is implemented for the barotropic primitive equation. The results show that the new method can reconstruct the initial conditions of the barotropic primitive equation conveniently and accurately and has a strong anti-noise characteristic.

[Keywords: Data assimilation, Barotropic primitive equation, Dual-number, Cost function].

Introduction

To effectively use the observation data to provide the dynamic optimal initial field for models, the initial data must be processed before the numerical weather prediction is made of the barotropic primitive equation model¹⁻². This is called data assimilation. Li et al.³ applied the 4-D data assimilation method to the data assimilation of the height field for the barotropic primitive equation and proved that this method can reduce the influence of observation errors. Wang et al.⁴ used the variational method to adjust the initial field of the barotropic primitive equation and the results demonstrate that this method can improve the forecast accuracy. Sun² used the method based on evolutionary strategy algorithm theory to do the data assimilation experiment of the barotropic primitive equation and improved its effectiveness. Currently, variational data assimilation (VDA) method is one of the most effective data assimilation methods in NWP⁵⁻⁸. The VDA methods⁹⁻¹³ include 3D and 4D VDA methods (3D / 4D-Var). The latter is the extension of the former in the time dimension and its performance is considerably better. 4D-Var is the most advanced method for ocean and weather forecasting and it has achieved great success in improving the accuracy of numerical prediction. However there still exists many difficulties. (i) Because 4D-Var needs to solve the adjoint model¹⁴⁻¹⁵, and the cost function is usually solved by using the

conjugate gradient method, the steepest descent method, the quasi-Newton iteration method and other iterative algorithms, the computation cost is particularly large. (ii) To reduce the computation cost, the incremental method is often used. But it cannot guarantee the convergence of results¹⁴⁻¹⁵. (iii) The hypothesis of the perfect model is implied in the assimilation window, this hypothesis is not hold when the model errors are large. (iv) The coding of the adjoint operator is a very complex and heavy task, and the parameterization of the physical process will lead to the discontinuity of the cost function.

In fact, the biggest difficulty in applying the VDA is the gradient calculation of the cost function. In addition to the adjoint method mentioned above, the difference method is often used to calculate the gradient value. But the choice of the difference step is a problem and the computation cost of the difference method is far greater than the adjoint method in the case of independent variables which have huge dimensions¹⁵. Additionally, the difference method is an approximation of spatial differentiation, so it exists truncation errors. Actually the gradient of the cost function is essentially a partial derivative of the cost function on the control variables, therefore, the key to solving the gradient of cost functions is to solving the partial derivative with high accuracy¹². While the dual-number has the high accuracy in solving the partial derivatives due to its special nature. The

dual-number concept and its theory have been proposed for more than 100 years, but dual-number was applied to robot kinematics, spatial structures and other kinematic and dynamic problems until the 1980s¹⁶⁻¹⁷. Zhang *et al.*¹⁸ applied the dual-number to the perspective N line problem, the advantages are that this method avoids the problem of the point correspondence and the errors. Zhu *et al.*¹⁹ described the synchronization control of relative motion for the spacecraft using dual-number and proved its effectiveness. Wang *et al.*²⁰ estimated the rotational motion and the translational motion of the chaser spacecraft relative to the target spacecraft based on the dual-number and the results showed that its can improve the estimate accuracy. In recent years, dual-number theory has opened up a new way for calculating the derivative information accurately and efficiently²¹⁻²². Cao, Song *et al.*²³ proposed a new data assimilation method based on dual-number theory, whose main idea is to convert the calculation process of the derivative into the calculation process of the dual-number field. And its effectiveness was verified in low-dimensional nonlinear dynamical systems and data assimilation problems of non-differentiable prediction models. Cao, song *et al.*²⁴ applied this method to calculate the finite-time Lyapunov exponent and proved that Lagrangian structures can be identified successfully. However, the dual-number theory has not been applied to the data assimilation of the barotropic primitive equation.

To overcome the shortcoming which the gradient of cost function is difficult to calculate in the data assimilation of the barotropic primitive equation, a data assimilation method based on dual-number theory is proposed in this paper. The new method can accurately obtain the cost function and its gradient without developing the tangent-linear/adjoint model.

Materials and Methods

The dual-number is defined as:

$$\hat{a} = a + \varepsilon a' \quad \dots (1)$$

Where a is the real part and a' is the dual part. Both a and a' are real numbers. The dual label ε does not indicate any specific value, but rather the nature of $\varepsilon \neq 0$ and $\varepsilon^n = 0$ ($n > 1$)²⁵⁻²⁶. For convenient expression, formula (1) can be written in the form of binary pair $\hat{a} = \langle a, \varepsilon a' \rangle$. The equal conditions of two dual-numbers are that their real and dual parts are all equal. The dual-number is 0, when the real part and

dual part are all 0. The module of a dual-number is defined as $|\hat{a}| = a$, which can be positive or negative. The conjugate number of a dual-number $\hat{a}$ is $\hat{a}^* = a - \varepsilon a'$; thus, $\hat{a}\hat{a}^* = a^2$ ²¹⁻²². If a and a' are extended to vector, a dual-number $\hat{a}$ can be extended to dual vectors.

Using the properties of the dual-number, the algebraic operations in the real number field can be extended to dual-number space. Assume that two dual-numbers are $\hat{x} = x + \varepsilon x'$ and $\hat{y} = y + \varepsilon y'$, and that their sum, difference and multiplication are²¹:

$$(x + x'\varepsilon) \pm (y + y'\varepsilon) = (x \pm y) + (x' \pm y')\varepsilon \quad \dots (2)$$

$$(x + x'\varepsilon) \times (y + y'\varepsilon) = xy + xy'\varepsilon + yx'\varepsilon + x'y'\varepsilon^2 = xy + (xy' + yx')\varepsilon \quad \dots (3)$$

In formula (3), $x'y'\varepsilon^2$ is omitted, not because x' and y' are too small but because the square of ε is 0. The dual-number multiplication represented by formula (3) is exactly equal, and there is no truncation error²⁶. Next considering the dual-number polynomial operation, formula (4) provides a polynomial representation in real number space:

$$P(x) = p_0 + p_1x + p_2x^2 + \dots + p_nx^n \quad \dots (4)$$

where $p_0, p_1, p_2, \dots, p_n$ are multinomial coefficients. Replacing the real number x with dual-number $\hat{x} = \langle x, x' \rangle = x + x'\varepsilon$ in formula (4) and simultaneously using the rules of dual-number addition and multiplication, it is easy to prove the formula as follows:

$$\begin{aligned} P(\hat{x}) &= p_0 + p_1(x + x'\varepsilon) + p_2(x + x'\varepsilon)^2 \\ &\quad + \dots + p_n(x + x'\varepsilon)^n \\ &= p_0 + p_1x + p_2x^2 + \dots + p_nx^n \\ &\quad + p_1x'\varepsilon + 2p_2xx'\varepsilon + \dots + np_nx^{n-1}x'\varepsilon \\ &= P(x) + P^{(1)}(x)x'\varepsilon \\ &= \langle P(x), P^{(1)}(x)x' \rangle \end{aligned} \quad \dots (5)$$

where $P^{(1)}$ represents the first-order derivative of the real polynomial $P(x)$ on a real variable x , x' is the seed number, which can take any value. Clearly, if $x' = 1$, then the dual part is exactly the value of the first-order derivative $dP(x)/dx$. The other basic algebras and new algorithms of standard functions in dual-number space are detailed in the literature²³.

Formula (5) can be further extended to the case where independent variable is the dual-number vector $\hat{x} = x + \varepsilon x'$:

$$f(\hat{x}) = f(x + \varepsilon x') = \langle f(x), \nabla f(x) \cdot x' \rangle \quad \dots (6)$$

In formula (6), x and $x' \in R^n$ are all multi-dimensional vectors. Therefore, as for the function $F(x)$, the directional derivative of direction $x' \in R^n$ at point x_0 can be derived by calculating the function in dual-number space directly, and the result is $\langle F(x_0), \nabla F(x_0) \cdot x' \rangle$.

The derivative value of independent variables can be obtained simultaneously when calculating the function value in dual-number space, which avoids the difference calculation. At the same time, truncation error will not be introduced because it avoids the difference operations, which results in the machine accuracy when it is implemented on a computer.

In the derivative computing method based on dual-number theory, for any cost function $J(x)$ that takes real variable x as the independent variable, the real variable x is first replaced with dual-number $\hat{x} = \langle x, x' \rangle = x + x'\varepsilon$. Then the dual function $\hat{J}(\hat{x})$ is expanded as Taylor series:

$$\begin{aligned} \hat{J}(\hat{x}) &= J(x) + \varepsilon x' \frac{J'(x)}{1!} + \varepsilon^2 x'^2 \frac{J''(x)}{2!} \frac{1}{2} \\ &+ \dots + (\varepsilon x')^n \frac{J^{(n)}(x)}{n!} + \dots \\ &= \langle J(x), (\nabla_x J) x' \rangle \end{aligned} \quad \dots (7)$$

Due to $\varepsilon^n = 0 (n > 1)$, high-order remainders that are greater than second order on the right-hand side of formula (7) can be omitted, and the real part and the dual part on both sides of the equation are strictly equal. If the value of x' is $x'=1$, then the value of $\hat{J}(\hat{x})$ is $\hat{J}(\hat{x}) = \langle J(x), (\nabla_x J) \rangle$. Therefore, if we want to calculate the first-order derivative of the cost function on the control variables, we only need to replace the independent variables in the cost function of the real number with the corresponding dual variables. After calculating the value of the function which is in dual-number form, the first-order derivative of the cost function on this independent variable can be obtained by taking its dual part²¹⁻²².

As for the multi-dimensional control vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$, obviously, there is an similar expression $\hat{J}(\hat{x}) = \langle J(x), (\nabla_x J) x' \rangle$.

The new method avoids the specialized derivative calculation, which can easily achieve the goal of

solving the gradient for the cost function in data assimilation. And the computation cost and errors are much less than the conventional difference method. After obtaining the cost function gradient of the initial height field, a suitable descent algorithm is selected to optimize the initial height field. The specific process of data assimilation for the barotropic primitive equation is as shown in Figure 1, and the specific steps are shown as follows¹⁹:

The first step: First, the conversation scheme of the second advection is used to achieve the numerical procedure for solving the barotropic primitive equation. Second, based on dual-number rules in dual-number space, the relevant procedures (including the equation of prediction model, the cost function and so forth.) are modified, which can be suitable for the numerical calculation of dual-number space.

The second step: Provide the initial guess $\hat{z}^0$ of the initial height field and construct the initial height field in dual-number form. The initial height field is constructed in the form of $\hat{z}^0 = (\langle z_1^0, e_1 \rangle, \langle z_2^0, e_2 \rangle, \dots, \langle z_n^0, e_n \rangle)$, where e_j represents the multi-dimensional unit vector whose value of the j th component is 1. And that the seed is 1 for each component of the initial height field.

The third step: Perform a forward integration for the barotropic primitive model using the provided guess value in dual-number space. The initial height field in dual-number form which is implemented in the second step is used as the initial condition of the

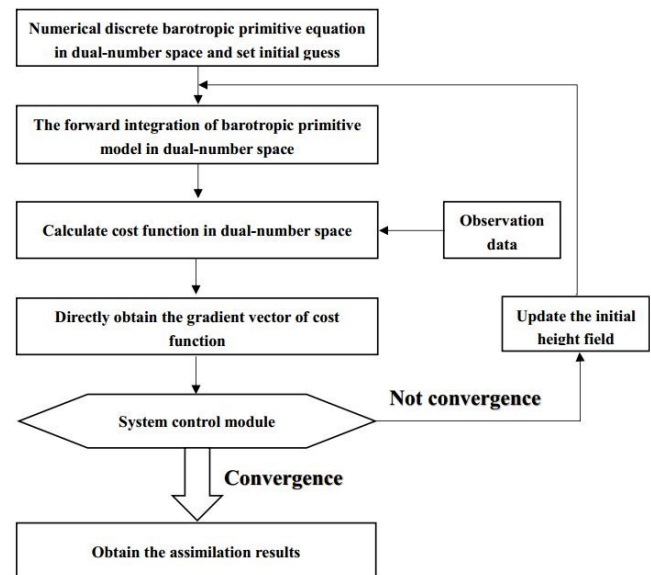


Fig. 1 — The flow diagram of data assimilation based on dual-number theory

barotropic primitive equation model, and the procedure implemented in the first step is used to perform a forward integration in dual-number space. In this way, the evolution track of system state $\hat{z}(t)$ in dual-number form will be obtained, where the real part and dual part of $\hat{z}(t)$ represent the value of the height field $z(t)$ and the derivative (or Jacobian matrix) of $z(t)$ on the initial height field z^0 at different time steps, respectively.

The forth step: Calculate the cost function and the gradient vector according to the dual-number theory. The value of cost function in dual-number form will be calculated by using $\hat{z}(t)$ and observation data $z^{obs}(t)$. Then the value of the cost function in real number form will be calculated according to formula (11), whose gradient of the initial height field component takes the same way to calculate.

The fifth step: The system control module determines whether the termination conditions are met. If the termination conditions of the procedure are not met, the system choose the optimal iterative step to adjust the initial height field. Moreover, a new iterative loop will start from the third step. If the termination conditions of the procedure are met, the procedure will stop, and the estimated value of the initial height field will be obtained at the same time. To reduce the cost of computation, this study uses the usual convergence criterion: the difference of the cost function gradient norm between two consecutive iterations is less than the pre-specified threshold $\|\nabla_{u'} J\| - \|\nabla_{u^{i+1}} J\| < 10^{-6}$. At such situations, the calculation errors are less than the machine errors¹².

Results and Discussion

For convenience, some finite difference operators are introduced (x direction):

$$\begin{aligned}\bar{F}^x &= \frac{1}{2}(F_{i+\frac{1}{2},j} + F_{i-\frac{1}{2},j}) \\ F_x &= \frac{1}{d}(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j}) \\ \bar{F}_x^x &= \frac{1}{2d}(F_{i+1,j} - F_{i-1,j})\end{aligned} \quad \dots (8)$$

In formula (8), F represents u, v, z and d is the grade space. Similarly, there are similar difference operators in the y direction. The barotropic primitive equation in the map projection is defined as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = -m(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) + f^* v - mg \frac{\partial z}{\partial x} = E \\ \frac{\partial v}{\partial t} = -m(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) + f^* u - mg \frac{\partial z}{\partial y} = G \\ \frac{\partial z}{\partial t} = -m^2[u \frac{\partial}{\partial x}(\frac{z}{m}) + v \frac{\partial}{\partial y}(\frac{z}{m}) + \frac{z}{m}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})] = H \end{cases} \quad \dots (9)$$

where $f^* = f + m^2[v\partial(1/m)/\partial x - u\partial(1/m)/\partial y]$, g is the acceleration of gravity. f is the Coriolis parameter. m is the amplification factor of map. z is the 500hPa geopotential height field. u and v are the component of the 500hPa horizontal wind velocity at x and y coordinate, respectively. E, G, H are the change tendency of u, v, z respectively. Using the conversation scheme of the second advection, the finite difference equation is as follows:

$$\begin{cases} \frac{\partial u_{i,j}}{\partial t} = -m_{i,j}(\bar{u}^x_x + \bar{u}^y_y + \bar{g}^x_x) + \tilde{f}^*_{i,j} v_{i,j} = E_{i,j} \\ \frac{\partial v_{i,j}}{\partial t} = -m_{i,j}(\bar{u}^x_x + \bar{v}^y_y + \bar{g}^y_y) - \tilde{f}^*_{i,j} u_{i,j} = G_{i,j} \\ \frac{\partial z_{i,j}}{\partial t} = -m_{i,j}^2[\bar{u}^x(\frac{z}{m})_x + \bar{v}^y(\frac{z}{m})_y + \frac{z_{i,j}}{m_{i,j}}(\bar{u}^x_x + \bar{v}^y_y)] = H_{i,j} \end{cases} \quad \dots (10)$$

where $\tilde{f}^* = f_{i,j} + u_{i,j} \bar{m}^y_y - v_{i,j} \bar{m}^x_x$.

In this experiment, it is assumed that an equilibrium relation between the wind field and the pressure field is satisfied. The initial values of the geostrophic wind are shown as follows:

$$\begin{cases} z_{i,j}|_{t=0} = z^0_{i,j} \\ u_{i,j}|_{t=0} = u^0_{i,j} = -\frac{m_{i,j}g}{f_{i,j}} \frac{\partial z^0_{i,j}}{\partial y} \\ v_{i,j}|_{t=0} = v^0_{i,j} = \frac{m_{i,j}g}{f_{i,j}} \frac{\partial z^0_{i,j}}{\partial x} \end{cases} \quad \dots (11)$$

For the prediction of the finite area, the horizontal lateral boundary conditions must be given artificially in the boundary. In this experiment, the fixed boundary conditions are used as boundary conditions:

$$\left. \frac{\partial u_{i,j}}{\partial t} \right|_{\beta} = \left. \frac{\partial v_{i,j}}{\partial t} \right|_{\beta} = \left. \frac{\partial z_{i,j}}{\partial t} \right|_{\beta} = 0 \quad \dots (12)$$

where β represents the lateral boundary of the forecast areas.

To maintain stability, the scheme of the time integration uses the frog-loop scheme and the difference scheme is as follows:

$$\begin{aligned} F_{i,j}^{n+\frac{1}{2}} &= F_{i,j}^n + \frac{\Delta t}{2} A_{i,j}^n \\ F_{i,j}^{n+1} &= F_{i,j}^n + 2\Delta t A_{i,j}^{n+\frac{1}{2}} \\ F_{i,j}^{n+2} &= F_{i,j}^n + 2\Delta t A_{i,j}^{n+1} \end{aligned} \quad \dots (13)$$

where A is the change tendency of F .

The calculation procedures are as follows²:

- (i) Input the 500hPa potential height field $z_{i,j}^0$.
- (ii) Calculate the map amplification factor $m_{i,j}$ and Coriolis parameter $f_{i,j}$.
- (iii) Calculate the initial values of the geostrophic wind $u_{i,j}^0$ and $v_{i,j}^0$.
- (iv) Calculate the change tendency of the wind field and the geopotential height field $E_{i,j}$, $G_{i,j}$ and $H_{i,j}$.
- (v) Perform the time integration and obtain the geopotential height field and the wind field of the forecast time.

The 500hPa initial geopotential height field is provided and the forecast is made by the barotropic primitive equation, the fixed boundary conditions and the conversation scheme of the second advection. The map projection is Lambert and Y axis is parallel to the 90 degree longitude. The X axis and the Y coordinate axis are perpendicular to each other. The grid number are $M=10, N=10$ and the horizontal distance is $d = 3 \times 10^5$.

In this experiment, the state vector is defined as $\mathbf{z} = (z_{(2,2)}, z_{(2,3)}, \dots, z_{(N-1,M-2)}, z_{(N-1,M-1)})^T$. The goal of data assimilation is to estimate the unknown initial state $\mathbf{z}_0 = (z_{0,(2,2)}, z_{0,(2,3)}, \dots, z_{0,(N-1,M-2)}, z_{0,(N-1,M-1)})^T$ from a certain number of observation data $\mathbf{z}_i^{obs} = (z_{i,(2,2)}^{obs}, z_{i,(2,3)}^{obs}, \dots, z_{i,(N-1,M-2)}^{obs}, z_{i,(N-1,M-1)}^{obs})^T$ for discrete distribution. In this experiment, $M = N = 10$.

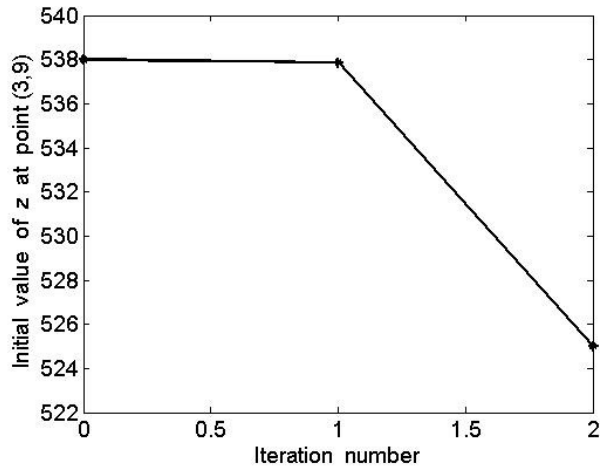
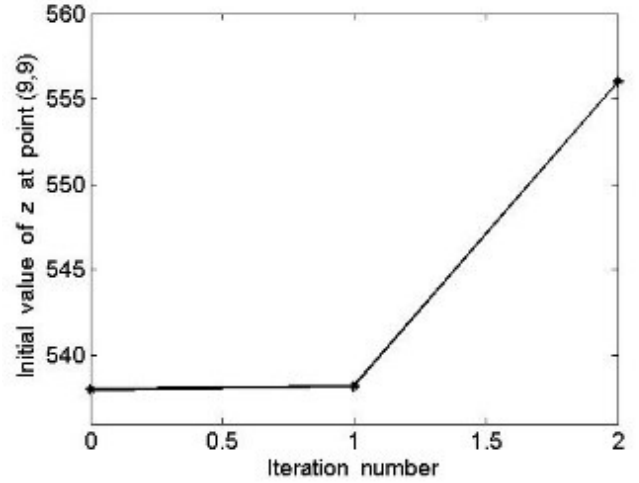
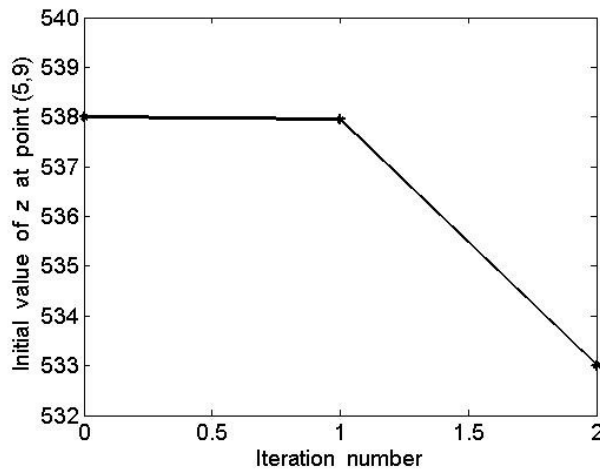
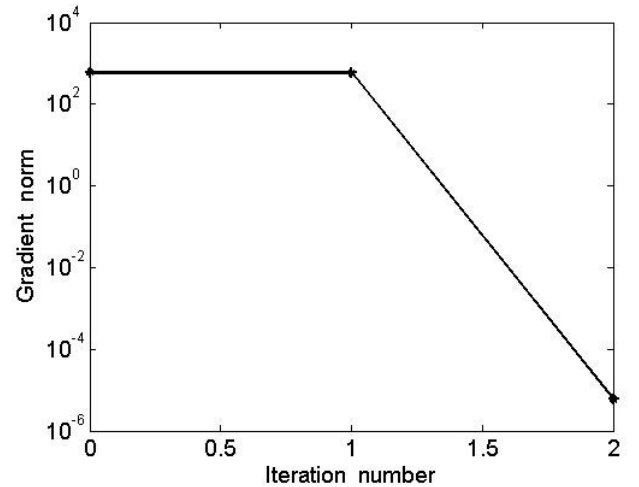
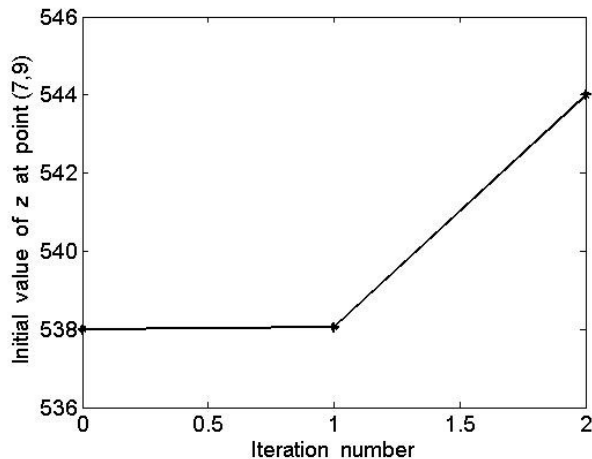
The conversation scheme of the second advection is first used to solve the barotropic primitive equation. Second, the numerical procedure for calculating the cost function is achieved. Third, its procedure is modified, which is based on dual-number rule in dual-number space. The acquisition process of the observation data is as follows: the initial state vector is $\mathbf{z}_0 = (z_{0,(2,2)}, z_{0,(2,3)}, \dots, z_{0,(N-1,M-2)}, z_{0,(N-1,M-1)})^T$.

Time interval is $[0, 0.1]$, and time step is $\Delta t = 0.001$. The space grid distance is $\Delta x = d = 3 \times 10^5$. The state value of $\mathbf{z}$ in discrete time series is derived by numerical integration and then Gaussian observation noise $N(0, \sigma_o)$ are superimposed to form the observation data. The mean value and standard deviation of the noises are 0 and σ_o respectively. If the background information is not considered, the cost function in discrete form is defined as follows [18]:

$$\hat{J}(\hat{\mathbf{z}}) = \frac{1}{2} \sum_{i=2}^{M-1} \sum_{j=2}^{N-1} \|\mathbf{z}_{i,j} - \mathbf{z}_{i,j}^{obs}\|^2 \quad \dots (14)$$

In formula (14), M and N represent the grid number in x direction and y direction respectively, $\|\cdot\|^2$ represents the Euclidean norm. The initial guess of the unknown initial state vector is $\mathbf{z}_0 = (538.0, 538.0, \dots, 538.0)$. In the process of the iterative estimation for the unknown initial state vector, to update the unknown vector $\mathbf{z}$ every time, the dual-number vector $\hat{\mathbf{z}}_i = (< z_{i,(2,2)}, \mathbf{e}_{1,(2,2)} >, < z_{i,(2,3)}, \mathbf{e}_{2,(2,3)} >, \dots, < z_{i,(M-1,N-2)}, \mathbf{e}_{(M-1,N-1)} >, < z_{i,(M-1,N-1)}, \mathbf{e}_{(M-1,N-1)} >)$ has to first be constructed, where $\mathbf{e}_{i,j}$ ($i = 2, 2, \dots, M-2; j = 2, 2, \dots, N-2$) represents the 64-D unit vector whose j th component is 1, and it is input into the numerical dual-number model of the barotropic primitive equation to make a forward ($0 \rightarrow T$) integration. Second, the cost function $\hat{J}(\hat{\mathbf{z}})$ is calculated in dual-number space. Third, the real part and the dual part are taken as the values of the cost function J and the gradient vector $\nabla_{\mathbf{z}} J$, respectively. Finally, the conjugate gradient algorithm is used to update the value of initial state vector [18]. To verify the effectiveness of the new method, the state variables from the 5th to the 10th time steps are selected as the observation data in the numerical experiment.

Figures 2-11 present the experiment results of data assimilation for the barotropic primitive equation without observation noise. Figures 2-5 present the change of the value for the initial wind field $z_{0,(3,9)}$, $z_{0,(5,9)}$, $z_{0,(7,9)}$ and $z_{0,(9,9)}$ in the iterative estimation process, respectively. The true values of the initial wind field $z_{0,(3,9)}$, $z_{0,(5,9)}$, $z_{0,(7,9)}$ and $z_{0,(9,9)}$ are 525, 533, 544 and 556, respectively. It can be observed from the figures that the iterative value converge to the true value, and the other sixty points have similar

Fig. 2 — The change of initial state $z_{(3,9)}$ with iteration numberFig. 5 — The change of initial state $z_{(9,9)}$ with iteration numberFig. 3 — The change of initial state $z_{(5,9)}$ with iteration numberFig. 6 — The change of the cost function $J(\mathbf{z})$ for the barotropicFig. 4 — The change of initial state $z_{(7,9)}$ with iteration number

results. To avoid redundancy, the results of the other sixty points are not given. Figure 6 shows the change of the cost function $J(\mathbf{z})$ with iteration number, and Figure 7 shows the change of the gradient normal $\|\nabla_{\mathbf{z}} J\|$ with iteration number. It can be observed from the figure that the cost function achieves convergence after the second iteration. Figure 8 and 9 show the comparison between the predicted pressure field after assimilation and the observed pressure field. Figure 10 and 11 show the comparison between the predicted wind field after assimilation and the observed wind field. It can be observed from the figures, the predicted field after assimilation, both the pressure wind and wind field, are consistent with the

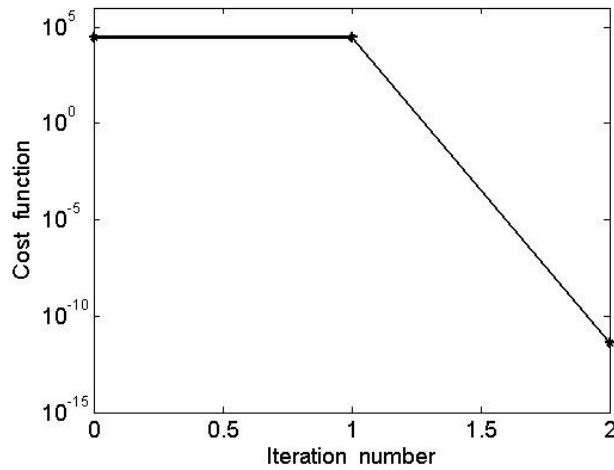


Fig. 7 — The change of the gradient norm $\|\nabla_z J\|$ for the barotropic primitive equation primitive equation

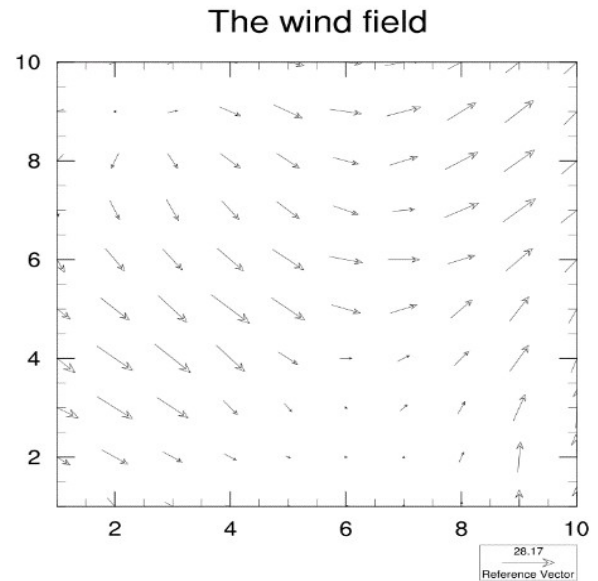


Fig. 10 — The wind field

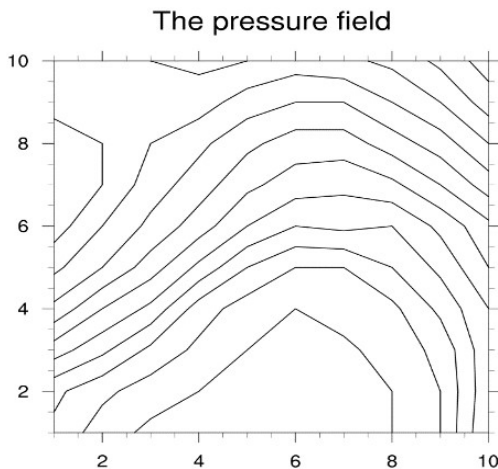


Fig. 8 — The pressure field

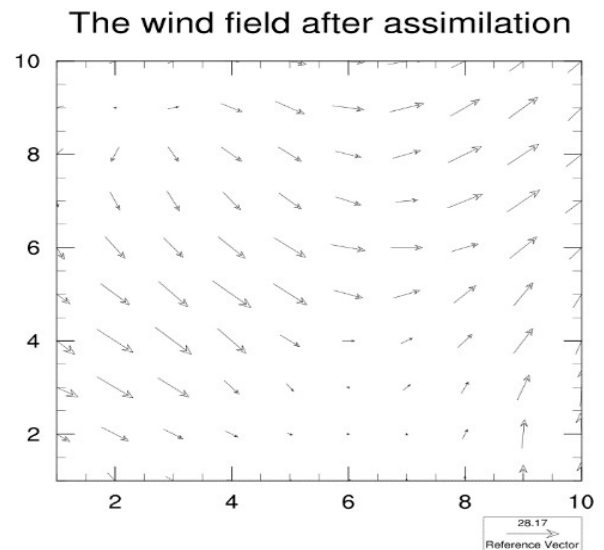


Fig. 11 — The wind field after assimilation

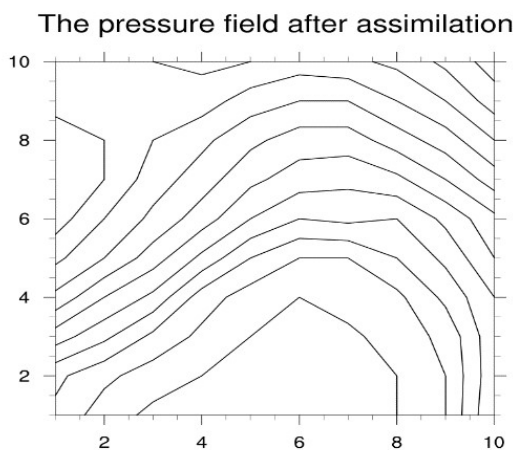


Fig. 9 — The pressure field after assimilation

observed field. The value of the initial wind field z is rounded to the 6th decimal. The results are shown in Table 1 (To avoid redundancy, only 16 points are listed). It can be seen from table 1 that the estimated value is very close to the true value.

Theoretically, when the gradient norm of cost function on the control variable is 0, the unknown initial state vector can take the optimal estimate. By analysing the experiment results, the following conclusions are obtained: in the case without background information and only using minimal

Table 1 — The results of data assimilation for the barotropic primitive equation under different levels of observation noises

	$Z_{0,(2,4)}$	$Z_{0,(2,7)}$	$Z_{0,(3,4)}$	$Z_{0,(3,7)}$	$Z_{0,(4,4)}$	$Z_{0,(4,7)}$	$Z_{0,(5,4)}$	$Z_{0,(5,7)}$	J	Iteration number
True value	520.0	519.0	523.0	519.0	526.0	522.0	535.0	524.0	\	
$\sigma_o = 0.0$	520.0000002	519.0000001	523.0000003	519.0000001	26.0000004	522.0000000	535.0000004	524.0000001	3.9E-12	2
$\sigma_o = 0.1$	519.9605814	518.9588112	523.0034181	519.0137226	525.9853722	521.9862364	535.0323795	523.9696858	1.38536	2
$\sigma_o = 0.2$	519.9320271	519.0201614	522.9667156	518.9260048	525.9808251	522.0172524	535.0146692	524.0354206	6.69285	2

	$Z_{0,(6,4)}$	$Z_{0,(6,7)}$	$Z_{0,(7,4)}$	$Z_{0,(7,7)}$	$Z_{0,(8,4)}$	$Z_{0,(8,7)}$	$Z_{0,(9,4)}$	$Z_{0,(9,7)}$	J	
True value	542.0	533.0	546.0	537.0	549.0	542.0	554.0	548.0	\	
$\sigma_o = 0.0$	542.0000001	533.0000001	546.0000003	536.9999999	549.0000002	541.9999999	553.9999999	547.9999997	3.9E-12	2
$\sigma_o = 0.1$	541.9772343	532.9962361	545.9671025	536.9767964	548.9439396	541.9791158	553.9601951	548.0565795	1.38536	2
$\sigma_o = 0.2$	541.9917912	533.0631917	545.8639226	537.0460205	548.9731755	542.0299784	554.0177101	547.926890	6.69285	2

observation information, the data assimilation method based on dual-number theory can accurately estimate the initial wind field of the barotropic primitive equation, and its estimation accuracy reaches $O(10^{-6})$

The advantages of the new assimilation method are that coding of the adjoint and tangent-linear models is no longer necessary and that the gradient of the cost function can be calculated very conveniently and accurately by using dual-number theory, whose value of the cost function can decline rapidly [18]. Table 1 shows the estimated value of the initial condition, the final value of the cost function and the iteration number for the barotropic primitive equation under different levels of observation noises. As shown in Table 1, when there are no observation errors, the estimation accuracy of the wind field is the highest. Moreover, the initial wind field value for each point can be accurate to the 6th decimal, which verifies the effectiveness of the data assimilation method based on dual-number theory in estimating the initial height field for the barotropic primitive equation. In addition, with increasing observation errors, the accuracy of data assimilation decreases, but iterative estimation results still converge to the true value. When the standard deviation of the observation errors is $\sigma_o = 0.2$, the results of data assimilation are still close to the true value, and their relative errors are less than 0.02%, which demonstrate that the new data assimilation method based on dual-number theory is capable of removing noises in the observations.

Conclusion

To provide an effective initial field for the barotropic primitive equation model, a new data assimilation method based on dual-number theory is proposed. By using dual-number theory, the process

of gradient analysis is transformed to the computation of the cost function in dual-number space, which can obtain the value of the gradient vector simply, efficiently and accurately. And the value of the cost function and its corresponding gradient vectors can be obtained simultaneously by only one forward computation in dual-number space. Additionally, the calculation precision is higher and the calculation cost is less. The results demonstrate that the new data assimilation method can effectively estimate the initial field of the barotropic primitive equation model and is capable of removing noises.

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